

A counterexample to monotonicity of relative mass in random walks

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Abstract

For a finite undirected graph $G = (V, E)$, let $p_{u,v}(t)$ denote the probability that a continuous-time random walk starting at vertex u is in v at time t . In this note we give an example of a Cayley graph G and two vertices $u, v \in G$ for which the function

$$r_{u,v}(t) = \frac{p_{u,v}(t)}{p_{u,u}(t)} \quad t \geq 0$$

is not monotonically non-decreasing. This answers a question asked by Peres in 2013.

1 Introduction

Let $G = (V, E)$ be a finite undirected regular graph. Let $p_{u,v}(t)$ denote the probability that a continuous-time random walk starting at vertex u is in v at time t . In this note we are interested in the function

$$r_{u,v}(t) = \frac{p_{u,v}(t)}{p_{u,u}(t)} \quad t \geq 0.$$

Clearly, in regular connected graphs for any $u \neq v$, we have $r_{u,v}(0) = 0$ and $\lim_{t \rightarrow \infty} r_{u,v}(t) = 1$. One might wonder if the function is monotonically non-decreasing. It is not difficult to see that there are regular graphs for which this is *not* the case. In fact, there are regular graphs such that $r_{u,v}(t) > 1$ for some vertices u, v and time t ; in particular, $r_{u,v}(t)$ is not monotonically non-decreasing. We give an example of such a graph in Appendix A. We thank Jeff Cheeger [Che15] for pointing this out to us.

For vertex-transitive graphs, however, it holds that $r_{u,v}(t) \leq 1$ for all vertices u, v and all $t \geq 0$. Indeed, using Cauchy-Schwarz and the reversibility of the walk,

$$\begin{aligned} p_{u,v}(t) &= \sum_{w \in V} p_{u,w}(t/2) p_{w,v}(t/2) \\ &\leq \left(\sum_w p_{u,w}(t/2)^2 \right)^{1/2} \cdot \left(\sum_w p_{w,v}(t/2)^2 \right)^{1/2} \\ &= p_{u,u}(t)^{1/2} \cdot p_{v,v}(t)^{1/2} = p_{u,u}(t). \end{aligned}$$

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This motivates the following question, asked in 2013 by Peres [Per13]:

Is the function $r_{u,v}$ monotonically non-decreasing in t for all vertex-transitive graphs and all vertices u, v ?

More recently, a special case of that question was asked independently by Price [Pri14]. Namely, Price asked whether for Brownian motion on flat tori (i.e., on \mathbb{R}^n modulo a lattice), it holds that for any point x , the density at x divided by the density at the starting point x_0 is monotonically non-decreasing in time. This would follow from a positive answer to Peres's question through a limit argument. Price gave a positive answer to his question for the case of a cycle ($n = 1$) and recently, a positive answer for arbitrary flat tori was found [RSD15]. This can be seen as further evidence for a positive answer to Peres's question.

In this note we give a negative answer to Peres's question. In fact, we do so through a Cayley graph.

Theorem 1.1. *There exists a Cayley graph $G = (V, E)$ and two vertices $u, v \in V$ such that the function $r_{u,v}$ is not monotonically non-decreasing.*

One remaining open question is whether $r_{u,v}$ is monotonically non-decreasing for *Abelian* Cayley graphs. The positive result of [RSD15] is a special case of that.

1.1 Some basic facts about continuous-time random walks

Given a weighted finite graph $G = (V, E)$ with weight function $w : E \rightarrow \mathbb{R}_+$ a continuous-time random walk $X = (X_t)_{t \geq 0}$ on G is defined by its heat kernel H_t , that at time $t > 0$ is equal to

$$H_t = e^{-t \cdot L},$$

where L is the Laplacian matrix of G given by $L_{u,v} = -w(u, v)$ for $u \neq v$, and $L_{u,u} = \sum_v w(u, v)$. As a result, for a random walk X starting at a vertex u the probability that X is in v at time t is equal to $p_{u,v}(t) := H_t(u, v)$. When G is a d -regular unweighted simple graph, we think of the edges as all having weight $1/d$, in which case the Laplacian of G is given by

$$L_{u,v} = \begin{cases} -1/d & \text{if } (u, v) \in E \\ 1 & \text{if } u = v \\ 0 & \text{otherwise.} \end{cases}$$

In this note we consider only vertex-transitive graphs, for which the sum $\sum_v w(u, v)$ is the same for all vertices u of the graph. Note that we do not insist that this sum is equal to 1, though this can be achieved by normalizing L , which corresponds to changing the speed of the random walk. For basic facts about continuous-time random walks see, e.g., [LPW09].

If G is a weighted Cayley graph with a generating set S and a weight function $w : S \rightarrow \mathbb{R}_+$, then a continuous-time random walk $X = (X_t)_{t \geq 0}$ on G is described by mutually independent Poisson processes of rate $w(g)$ for each group generator $g \in S$, where each process indicates the times when X jumps along the corresponding edge.

2 Non-monotonicity of time spent at the origin in the hypercube graph

For an integer $d \geq 1$ denote by Q_d the d -dimensional hypercube graph. The vertices of Q_d are $\{0,1\}^d$ and there is an edge between two vertices u and v if and only if they differ in exactly one coordinate. Let $X = (X_t)_{t \geq 0}$ be a continuous-time random walk on Q_d starting at the origin, denoted by $\mathbf{0} = (0, \dots, 0) \in \{0,1\}^d$. Denote by $C_d(t)$ the expected time spent at the origin until time t , conditioned on the event that $X_t = \mathbf{0}$. That is,

$$C_d(t) = \int_0^t \Pr[X_s = \mathbf{0} | X_t = \mathbf{0}] ds.$$

In this section we show that for d sufficiently large $C_d(t)$ is not monotonically non-decreasing.

Lemma 2.1. *Let $d \in \mathbb{N}$ be sufficiently large. Then, there are some $t_1 < t_2$ such that $C_d(t_1) > C_d(t_2)$, and in particular, the function C_d is not monotonically non-decreasing in t .*

Remark. Numerically, one can see that the function C_d is not monotone for $d \geq 5$. See Figure 1. Since C_d has a closed form expression (as can be seen from the calculations below), one can probably show non-monotonicity directly for C_5 by analyzing the function, though doing so would likely be messy and not too illuminating.

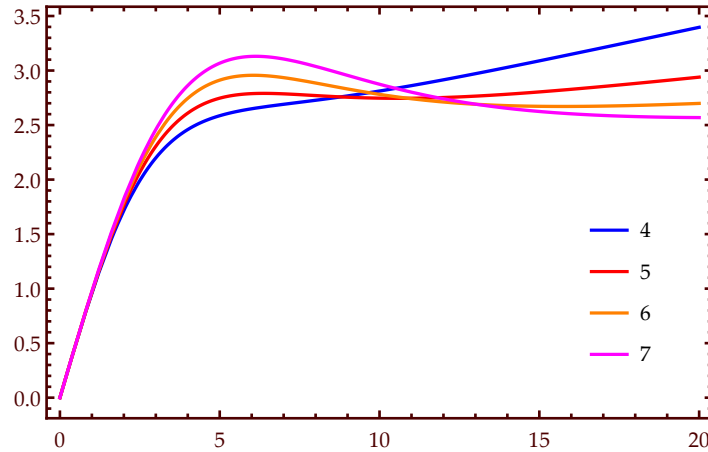


Figure 1: $C_d(t)$ for $d = 4, 5, 6, 7$ (from top right to bottom right).

Before proving Lemma 2.1 we prove the following claim.

Claim 2.2. *Let $d \geq 1$, and let Q_d be the d -dimensional hypercube graph. Let $X = (X_t)_{t \geq 0}$ be a continuous-time random walk on Q_d starting at $\mathbf{0}$. Then,*

$$\Pr[X_t = \mathbf{0}] = \left(\frac{1 + e^{-2t/d}}{2} \right)^d.$$

Proof. Since X moves in each coordinate with rate $1/d$, it follows that for each $i \in [d]$ the number of steps in direction i up to time t is distributed like $\text{Pois}(t/d)$. Therefore,

$$\Pr[(X_t)_i = 0] = \Pr[\text{Pois}(t/d) \text{ is even}] = (1 + e^{-2t/d})/2,$$

where we used that the probability that $\text{Pois}(\lambda)$ is even is

$$\Pr[\text{Pois}(\lambda) \text{ is even}] = e^{-\lambda} \cdot \sum_{j \text{ even}} \frac{\lambda^j}{j!} = e^{-\lambda} \cdot \frac{1}{2} \left(\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} + \sum_{j=0}^{\infty} \frac{(-\lambda)^j}{j!} \right) = e^{-\lambda} \cdot \frac{1}{2} (e^{\lambda} + e^{-\lambda}) .$$

Since the coordinates of X move independently the result follows. \square

We now prove Lemma 2.1.

Proof of Lemma 2.1. We show below that for all $d \geq 1$, it holds that

1. $C_d(\sqrt{d}) \geq e^{-1}\sqrt{d}$,
2. $C_d(d) \leq 6$.

This clearly proves the lemma for d sufficiently large.

To prove Item 1, we show that if a walk starting from the origin is at the origin at time \sqrt{d} , then with constant probability it stayed at the origin throughout that time interval. Intuitively, this is because the probability of a coordinate flipping twice during that time is of order only $1/d$ and so with constant probability none of the d coordinates flips. In more detail, by Claim 2.2,

$$\Pr[X_{\sqrt{d}} = \mathbf{0}] = \left(\frac{1 + e^{-2/\sqrt{d}}}{2} \right)^d \leq \left(1 - \frac{1}{\sqrt{d}} + \frac{1}{d} \right)^d = \left(1 - \frac{\sqrt{d}-1}{d} \right)^d \leq e^{-\sqrt{d}+1} ,$$

where we used the inequality $e^{-x} \leq 1 - x + x^2/2$ valid for all $x \geq 0$. On the other hand, by definition of a continuous-time random walk the probability that X stays in $\mathbf{0}$ during the entire time interval $[0, \sqrt{d}]$ is equal to $\Pr[X_{[0, \sqrt{d}]} \equiv \mathbf{0}] = e^{-\sqrt{d}}$. Therefore,

$$\Pr[X_{[0, \sqrt{d}]} \equiv \mathbf{0} | X_{\sqrt{d}} = \mathbf{0}] \geq e^{-1} ,$$

and hence the expected time spent at the origin conditioned on $X_{\sqrt{d}} = \mathbf{0}$ is as claimed in Item 1.

We next prove Item 2. Intuitively, here there is enough time for coordinates to flip twice, and only a very small part of the time will be spent at the origin. By definition of C_d and Claim 2.2 we have

$$\begin{aligned} C_d(t) &= \int_0^t \frac{\Pr[X_s = \mathbf{0}] \cdot \Pr[X_{t-s} = \mathbf{0}]}{\Pr[X_t = \mathbf{0}]} ds \\ &= \int_0^t (h_d(t, s))^d ds, \end{aligned}$$

where

$$h_d(t, s) = \frac{(1 + e^{-2s/d})(1 + e^{-2(t-s)/d})}{2(1 + e^{-2t/d})} = \frac{1 + e^{-2s/d} + e^{-2(t-s)/d} + e^{-2t/d}}{2(1 + e^{-2t/d})} .$$

Since $h_d(t, s)$ is convex as a function of s , for all $0 \leq s \leq t/2$ we have $h_d(t, s) \leq \ell(s)$ where ℓ is the unique linear function satisfying $\ell(0) = h_d(t, 0)$ and $\ell(t/2) = h_d(t, t/2)$. Therefore, taking $t = d$, we get

$$h_d(d, s) \leq \ell(s/d) = 1 - \frac{cs}{d} ,$$

where $c = (1 - e^{-1})^2 / (1 + e^{-2})$. Noting that $h_d(t, s) = h_d(t, t - s)$, we get

$$C_d(d) = \int_0^d (h_d(d, s))^d ds = 2 \int_0^{d/2} (h_d(d, s))^d ds < 2 \int_0^{d/2} \left(1 - \frac{cs}{d}\right)^d ds < 2 \int_0^{d/2} e^{-cs} ds \leq \frac{2}{c}.$$

This completes the proof of Lemma 2.1. \square

3 Proof of Theorem 1.1

In this section we prove Theorem 1.1. We first give a proof for a weighted graph, and then remark on how to convert it into an unweighted graph. For $d \in \mathbb{N}$ sufficiently large we define the weighted graph G to be the *lamplighter graph* on Q_d , whose edges corresponding to steps on Q_d are of weight $1/d$, and edges corresponding to toggling a lamp are of weight ε , for some $\varepsilon > 0$ sufficiently small that depends on d and t_1, t_2 from Lemma 2.1.

In more detail, the weighted lamplighter graph G is described by placing a lamp at each vertex of Q_d and a lamplighter walking on Q_d . A vertex of G is described by the location $x \in \{0, 1\}^d$ of the lamplighter, and a configuration $f : \{0, 1\}^d \rightarrow \{0, 1\}$ indicating which lamps are currently on. In each step the lamplighter either makes a step in the graph Q_d or toggles the state of the lamp in the current vertex. More formally, we have an edge between (x, f) and (y, g) if and only if either

1. $(x, y) \in E_d$ and $f = g$ (this corresponds to a step in Q_d) or
2. $x = y$ and f and g differ on the input x and are equal on all other inputs (this corresponds to toggling a lamp at x).

The weights of the edges of the first type are $1/d$, and the edges of the second type are of weight ε . Thus, in a random walk on G , the steps of the lamplighter are distributed as in a random walk on Q_d , and the number of times the lamps are toggled in a time interval of length T is distributed like $\text{Pois}(\varepsilon T)$ independently of the lamplighter's walk. It is well known that the lamplighter graph is a Cayley graph (see, e.g., [PR04]).

Let u be the vertex in G corresponding to the lamplighter being at the origin with all lights off. Let v be the vertex in G corresponding to the lamplighter being at the origin with the light at the origin being on, and all other lights off. We show below that $r_{u,v}$ is not monotonically non-decreasing. More specifically, we show that $r_{u,v}(t_1) > r_{u,v}(t_2)$, where $t_1 < t_2$ are from Lemma 2.1.

Let $X = (X_t)_{t \geq 0}$ be a continuous-time random walk on G starting at $X_0 = u$. Denote by Y_t the number of times a toggle occurred during the time interval $[0, t]$. Denote by $Z = (Z_t)_{t \geq 0}$ the trajectory of the lamplighter, i.e., the projection of X to the first coordinate. Note that by definition Z is a continuous-time random walk on Q_d , and that Z is independent of Y_t .

Claim 3.1. *Let $u, v \in V$ be as above. Then, for all $t > 0$ it holds that*

$$0 \leq p_{u,u}(t) - e^{-\varepsilon t} \cdot \Pr[Z_t = \mathbf{0}] \leq \varepsilon^2 t^2, \quad (1)$$

and

$$0 \leq p_{u,v}(t) - \varepsilon e^{-\varepsilon t} \cdot C_d(t) \cdot \Pr[Z_t = \mathbf{0}] \leq \varepsilon^2 t^2. \quad (2)$$

Using the claim,

$$r_{u,v}(t) = \frac{p_{u,v}(t)}{p_{u,u}(t)} = \varepsilon \cdot C_d(t) \pm O(\varepsilon^2),$$

where $O(\cdot)$ hides a constant that depends on d and t . In particular, for $t_1 < t_2$ from Lemma 2.1, and $\varepsilon > 0$ sufficiently small we get that $r_{u,v}(t_1) > r_{u,v}(t_2)$, which proves Theorem 1.1.

Intuitively, (1) holds because the probability of toggling a lamp twice is very small, and hence $p_{u,u}(t)$ is approximately equal to the probability that no lamp has changed its state multiplied by the probability that a random walk on Q_d will be at the origin at time t . The intuition for (2) is that in order to get from u to v , in addition to getting back to the origin, the lamplighter must toggle the switch while being at the origin, and the probability of that is roughly $\varepsilon \cdot C_d(t)$.

Proof of Claim 3.1. For $p_{u,u}$ we have

$$p_{u,u} = \Pr[X_t = u \wedge Y_t = 0] + \Pr[X_t = u \wedge Y_t \geq 2].$$

Since Y_t is distributed like $\text{Pois}(\varepsilon t)$, the second term satisfies

$$0 \leq \Pr[X_t = u \wedge Y_t \geq 2] \leq \Pr[Y_t \geq 2] \leq \varepsilon^2 t^2,$$

and for the first term, by independence between Y_t and Z_t we have

$$\Pr[X_t = u \wedge Y_t = 0] = \Pr[Z_t = \mathbf{0} \wedge Y_t = 0] = e^{-\varepsilon t} \cdot \Pr[Z_t = \mathbf{0}],$$

proving (1).

For $p_{u,v}$ we similarly have

$$p_{u,v}(t) = \Pr[X_t = v \wedge Y_t = 1] + \Pr[X_t = v \wedge Y_t \geq 2].$$

As above, the second term is at most $\varepsilon^2 t^2$. For the first term, let E_t be the event that $Y_t = 1$, and the unique lamp that is on at time t is the lamp at the origin. Denote by T_0 the time spent by Z at the origin in the time interval $[0, t]$. Then, conditioning on Z , the event E_t holds if and only if a unique switch happened during T_0 time, and zero switches in the remaining time. Therefore, by independence of a Poisson process in disjoint intervals

$$\Pr[E_t|Z] = \Pr[\text{Pois}(\varepsilon T_0) = 1|Z] \cdot \Pr[\text{Pois}(\varepsilon(t - T_0)) = 0|Z] = \varepsilon T_0 \cdot e^{-\varepsilon T_0} \cdot e^{-\varepsilon(t - T_0)} = \varepsilon e^{-\varepsilon t} \cdot T_0.$$

This implies that

$$\Pr[X_t = v \wedge Y_t = 1] = \Pr[E_t|Z_t = \mathbf{0}] \cdot \Pr[Z_t = \mathbf{0}] = \varepsilon e^{-\varepsilon t} \cdot \mathbb{E}[T_0|Z_t = \mathbf{0}] \cdot \Pr[Z_t = \mathbf{0}].$$

Therefore, since $C_d(t) = \mathbb{E}[T_0|Z_t = \mathbf{0}]$ we get (2), and the claim follows. \square

Converting G into an unweighted graph. Below we show how to convert a weighted Cayley graph G into an unweighted one, while preserving the property in Theorem 1.1. Let (G, S_G) be a weighted Cayley graph with the generating set $S_G = \{g_1, \dots, g_k\}$, and suppose that the weights $w : S_G \rightarrow \mathbb{R}_+$ are integers for all $g \in S_G$. For $N \in \mathbb{N}$ sufficiently large define the graph H by replacing each vertex $v \in G$ with an N -clique $\{(v, i) : i \in \mathbb{Z}_N\}$, and replacing each edge (u, ug) in G of weight $w(g)$ with $w(g)$ perfect matchings $\{(u, i), (ug, i + j) : i \in \mathbb{Z}_N\}_{j=1}^{w(g)}$.

Formally, the vertices of the graph H are $G \times \mathbb{Z}_N = \{(v, i) : v \in G, i \in \mathbb{Z}_N\}$, and the set of generators S_H for the Cayley graph on H is given by

$$S_H = \{(0, i) : i \in \mathbb{Z}_N \setminus \{0\}\} \cup \bigcup_{g \in S_G} \{(g, j) : j \in \{1, \dots, w(g)\}\}.$$

Note that the projection of a continuous-time random walk on H to the first coordinate is a random walk on G , slowed down by $\deg(H)$. Moreover, assuming N is larger than, say, $\sum_{g \in S_G} w(g)$, after constant time the two coordinates become close to independent with the second coordinate being uniform. Therefore, if u, v are vertices in G , and $x = (u, 0), y = (v, 0)$ are the corresponding vertices in H , then for any time $t > 0$ and $t' = \deg(H) \cdot t$ it holds that $p_{x,y}(t') = \frac{1}{N}(p_{u,v}(t) \pm o_N(1))$ and hence $r_{x,y}(t') = r_{u,v}(t) \pm o_N(1)$.

For the graph G given in the proof of Theorem 1.1 above, we may assume that $1/\varepsilon$ is an integer, and so, by multiplying all weights by d/ε we get a Cayley graph with integer weights. Hence, by applying the foregoing transformation we get a simple unweighted Cayley graph H for which $r_{u,v}$ is not monotonically non-decreasing for some $u, v \in H$.

A Appendix: A counterexample in a regular non-transitive graph

Below we give a simple example of a regular non-transitive graph such that $r_{u,v}(t) > 1$ for some vertices u, v and some time t ; in particular, $r_{u,v}(t)$ is not monotonically non-decreasing, since $r_{u,v}(t) \rightarrow 1$ as $t \rightarrow \infty$. We thank Jeff Cheeger [Che15] for pointing this out to us.

Proposition A.1. *Let L be the Laplacian of a regular graph on vertex set V . Denote its eigenvalues by $0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{|V|}$ and by $f_i \in \mathbb{R}^V$ the corresponding normalized eigenvectors. Suppose that $0 < \lambda_2 < \lambda_3$, and that f_2 is such that $f_2(v) > f_2(u) > 0$ for some vertices u, v . Then, there is some $t > 0$ such that $r_{u,v}(t) > 1$.*

Proof. Let $\pi_u \in \mathbb{R}^V$ be the vector with $\pi_u(u) = 1$ and $\pi_u(u') = 0$ for all $u' \neq u$. Writing $\pi_u = \sum \alpha_i f_i$ for $\alpha_i = \langle \pi_u, f_i \rangle = f_i(u)$, for all $w \in V$ we have

$$e^{-tL}\pi_u(w) = \sum_{i=1}^{|V|} e^{-t\lambda_i} \alpha_i \cdot f_i(w) = c + e^{-\lambda_2 t} f_2(u) f_2(w) + O(e^{-\lambda_3 t}),$$

where $O()$ hides some constants that may depend on the graph, but not on t , and $c = \alpha_1 \cdot f_1(w)$ is independent of w since f_1 is a constant function. Using the facts that $f_2(v) > f_2(u) > 0$ and $\lambda_3 > \lambda_2$, it follows that for sufficiently large t ,

$$r_{u,v}(t) = \frac{e^{-tL}\pi_u(v)}{e^{-tL}\pi_u(u)} > 1,$$

as desired. □

Graphs satisfying the constraints in Proposition A.1 are in abundance. As a concrete example, consider the 4-regular graph on 10 vertices shown in Figure 2. Using Mathematica, we see that the second eigenvalue of the Laplacian of this graph is $\lambda_2 = \frac{1}{8}(7 - \sqrt{17}) \approx 0.36$, and it is a simple eigenvalue. The corresponding (non-normalized) eigenvector with vertices ordered from left to right is $(c, 1, 1, 1, 1, -1, -1, -1, -1, -c)$, where $c = 3 - \frac{1}{2}(7 - \sqrt{17}) \approx 1.56$. In particular, Proposition A.1 is applicable to this graph.

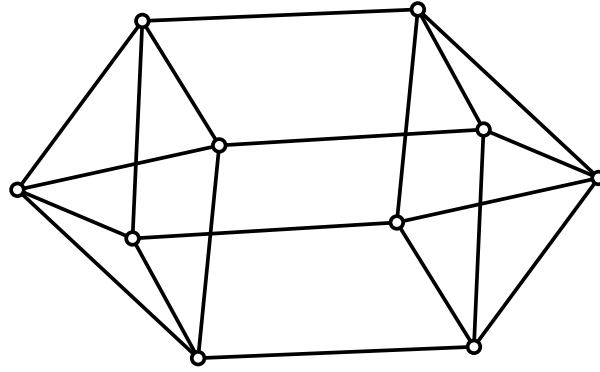


Figure 2: A cube with two square pyramids attached.

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